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Quantum fluctuations and glassy behavior: The case of a quantum particle in a random potential

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In this paper we expand our previous investigation of a quantum particle subject to the action of a random potential plus a fixed harmonic potential at a finite temperature T. In the classical limit the system reduces to a well-known "toy" model for an interface in a random medium. It also applies to a single quantum particle such as an electron subject to random interactions, where the harmonic potential can be tuned to mimic the effect of a finite box. Using the variational approximation or, alternatively, the limit of large spatial dimensions, together with the use of the replica method, we are able to solve the model and obtain its phase diagram in the $T-(\hbar^2/m)$ plane, where m is the particle's mass. The phase diagram is similar to that of a quantum spin glass in a transverse field, where the variable $\hbar^2/m$ plays the role of the transverse field. The glassy phase is characterized by replica symmetry breaking. The quantum transition at zero temperature is also discussed.

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I. INTRODUCTION

The effects of quantum fluctuations on phase transitions are a topic of significant current research. On the other hand, the transition into a glassy phase in disordered systems is a topic that is far from trivial and requires special analytical and numerical techniques. In particular, the behavior of interfaces in disordered systems has been the subject of many recent papers [1–12]. It is of interest to investigate the combined effect of disorder and quantum fluctuations that play a significant role at low temperatures.

In a recent paper [13], we have focused on the problem of a quantum particle in a random potential with power law correlations plus a fixed harmonic restoring force. This problem has previously been studied classically [1,8,9,11,12] by using the variational approximation [8,9,11] (or, alternatively, the large-N limit [10,12], where N is the number of dimensions). A transition into a glassy phase has been identified. The signature of such a transition is the appearance of solutions with replica symmetry breaking (RSB) [14]. RSB is typically associated with a complicated free-energy landscape characterized by many local minima that are separated by high barriers. A sharp transition for a single particle must be an artifact of the variational approximation (or, alternatively, the large-N limit). In reality this must be a crossover effect over a range of temperatures.

The "toy" model of a particle in a random potential in one dimension is the simplest example of an interface problem where the location of the particle stands for the location of an interface between to phases (such as up and down spins) and the random potential stands for quenched impurities that can pin the interface. Understanding this problem is the first step of investigating higher-dimensional manifolds. But the quantum version of the model also applies directly to a particle such as an electron in a dirty metal [15], where the strength of the harmonic potential can be tuned to mimic a finite box that the electron is confined to, like the situation that occurs in certain mesoscopic systems.

The question we have had in mind was what the effects of quantum fluctuations on this glassy transition were. There are some similarities between this problem and the quantum spin-glass problem in a transverse field [16]. In that problem, tunneling effects produced by the transverse field eventually lead, for strong enough field, to the destruction of the spin-glass phase. But this phase, when it exists, is still characterized by RSB for the infinite range model. With this analogy in mind, one should notice, though, that the transition that has been found for a particle in a random potential [8,9,11] is of the Almeida-Thouless type [17] in the sense that it is associated with the appearance of RSB but not with an order-disorder (spin-glass-like) transition.

In this paper we expand our previous investigation of the quantum particle and also correct a typographical er-

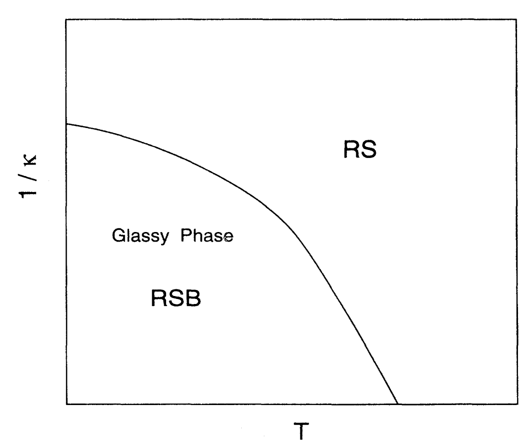

FIG. 1. Schematic phase diagram of a quantum particle in a random potential plus a fixed harmonic potential.

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ror that unfortunately occurred in Eq. (7) of [13], propagated into Eq. (8), and affected some of the numerical results. We find as before that the glassy phase, characterized by RSB, persists in the presence of quantum fluctuations. However, we find that for small enough particle mass [m and $\hbar$ enter only in the combination $\kappa = (m/\hbar^2)$], the glassy phase ceases to exist. Thus the parameter $1/\kappa$ plays a similar role to the transverse field in the Ising spin glass. A schematic phase diagram is depicted in Fig. 1.

II. BASIC DEFINITIONS

The density matrix for a particle at finite temperature $T = 1/k_B\beta$, subject to a harmonic potential and a random potential V, is given by the functional integral [18]

$$\rho(\mathbf{x},\mathbf{x}',U) = \int_{\mathbf{x}(0)=\mathbf{x}}^{\mathbf{x}(U)=\mathbf{x}'} [d\mathbf{x}] \exp\left\{ -\frac{1}{\hbar}\int_0^U \left[\frac{m[\dot{\mathbf{x}}(u)]^2}{2} + \frac{\mu[\mathbf{x}(u)]^2}{2} + V(\mathbf{x}(u)) \right] du \right\}, \tag{2.1}$$

where $\mathbf{x}$ is an N-dimensional vector (N is the number of spatial dimensions) and $U = \beta\hbar$. The variable u has dimensions of time and is often referred to as the Trotter dimension. For the diagonal elements of ρ we observe that the trajectory $\mathbf{x}(u)$ forms a closed path. In this paper we are concerned with a random quenched potential $V(x)$ and quantities of interest are the averaged free energy and the mean-square displacement given by

$$-\beta\langle F\rangle_R = \left\langle \ln \int \rho(\mathbf{x},\mathbf{x}) d\mathbf{x} \right\rangle_R , \tag{2.2}$$

$$\langle\langle \mathbf{x}^2 \rangle\rangle_R = \left\langle \frac{\int \rho(\mathbf{x},\mathbf{x})\mathbf{x}^2 d\mathbf{x}}{\int \rho(\mathbf{x},\mathbf{x})\, d\mathbf{x}} \right\rangle_R , \tag{2.3}$$

where $\langle\,\rangle_R$ stands for an average over the random realizations of the potential. The difficulty in carrying out the quenched averages stems from the fact that in Eq. (2.2) the average is taken after taking the logarithm, i.e., the averaged free energy is not the logarithm of the averaged partition function. Similarly, in Eq. (2.3) the average of the quotient is not equal to the quotient of the averages. We take the potential $V(x)$ to be Gaussian distributed, which means that the probability for a given realization of the potential is

$$P(V(x)) = C \exp\left(-\int dx dx' V(x) \Delta(x - x') V(x') \right), \tag{2.4}$$

with some known function $\Delta(x - x')$. It is thus sufficient to know only the first two moments of the distribution, viz.,

$$\langle V(\mathbf{x})\rangle_R = 0,\ \langle V(\mathbf{x})V(\mathbf{x}')\rangle_R = -Nf\left(\frac{(\mathbf{x}-\mathbf{x}')^2}{N} \right), \tag{2.5}$$

where the functions f and Δ are related to each other. The function f describing the correlations of the random potential is taken to decay as a power at large distances:

$$f(y) = \frac{g}{2(1-\gamma)} (a_0 + y)^{1-\gamma}. \tag{2.6}$$

The index γ describes the behavior of the correlations of the disorder at large distances. In this paper we consider only two cases: that of $\gamma = 3/2$, which we call the case of short-range correlations, and that of $\gamma = 1/2$, which we call the case of long-range correlations [7,8,11]. The parameter a_0 plays the role of a short-distance regulator for f. We chose these values of γ in order to make contact with known results in the classical case. In the classical case it has been shown (for $N = 1$) that correlations falling faster than those characterized by $\gamma = 3/2$, even correlations falling exponentially fast, are equivalent within the variational approximation to the case of $\gamma = 3/2$ [7,11]. This fact also holds in the quantum case, as demonstrated in Sec. III below. The long-range case is of interest because of its connection with the directed-polymer problem and with the random-field Ising model [6,7,9,11].

We apply the replica trick in order to carry out the quenched average over the random realizations. We consider n copies of the system and obtain for the averaged density matrix

$$\rho(\mathbf{x}_1\cdots\mathbf{x}_n,\mathbf{x}_1\cdots\mathbf{x}_n,U) = \int_{\mathbf{x}_a(0)=\mathbf{x}_a}^{\mathbf{x}_a(U)=\mathbf{x}_a} \prod_{a=1}^{n} [d\mathbf{x}_a] \exp\{-\mathcal{H}_n/\hbar\}, \tag{2.7}$$

$$\mathcal{H}_n = \frac{1}{2}\int_0^U du \sum_a \left[m\dot{\mathbf{x}}_a^2(u) + \mu\mathbf{x}_a^2(u) \right] + \frac{1}{2\hbar}\int_0^U du \int_0^U du' \sum_{a,b} N \times f\left(\frac{[\mathbf{x}_a(u)-\mathbf{x}_b(u')]^2}{N} \right). \tag{2.8}$$

Note that this equation, although somewhat similar, is quite different from the equation for the n-body Hamiltonian corresponding to the directed polymer problem in $1 + N$ dimensions [7]. The difference is, first, the fact that the integral over the "time" variable u is on a finite interval, and second and more significantly, that the n-body potential is nonlocal in time, i.e., it involves a double integral over both u and u'.

III. THE LARGE-N LIMIT AND THE VARIATIONAL APPROXIMATION

In the large-N limit, we introduce collective variable fields $r_{ab}(u,u')$,

$$r_{ab}(u,u') = \frac{1}{N}\mathbf{x}_a(u)\cdot\mathbf{x}_b(u') \tag{3.1}$$

and Lagrange multiplier fields $s_{ab}(u,u')$ to implement (3.1). We find [10]

$$\rho(\mathbf{x}_1\cdots\mathbf{x}_n,\mathbf{x}_1\cdots\mathbf{x}_n,U) = \int_{\mathbf{x}_a(0)=\mathbf{x}_a}^{\mathbf{x}_a(U)=\mathbf{x}_a}\prod_{a=1}^{n}[d\mathbf{x}_a]\int\prod_{a\leq b}[dr_{ab}(u,u')][ds_{ab}(u,u')]$$
$$\times\exp\left\{-\frac{1}{2\hbar^2}\int_0^U du\int_0^U du'\sum_{a,b}[Nr_{ab}(u,u')-\mathbf{x}_a(u)\cdot\mathbf{x}_b(u')]s_{ab}(u,u')\right.$$
$$-\frac{1}{2\hbar}\int_0^U du\sum_a\left[m\dot{\mathbf{x}}_a^2(u)+\mu\mathbf{x}_a^2(u)\right]$$
$$\left.-\frac{1}{2\hbar^2}\int\int_0^U dudu'\sum_{a,b}Nf(r_{aa}(u,u)+r_{bb}(u',u')-2r_{ab}(u,u'))\right\}. \tag{3.2}$$

From here it follows that the free energy is given by

$$n\beta\langle F\rangle_R/N=\frac{1}{2\hbar^2}\int_0^U du\int_0^U du'\sum_{a,b}^{n}[r_{ab}(u,u')s_{ab}(u,u')$$
$$+f(r_{aa}(u,u)+r_{bb}(u',u')-2r_{ab}(u,u'))]-\ln\int_{-\infty}^{\infty}\prod_a dx_a\int_{x_a(0)=x_a(U)=x_a}\prod_a[dx_a]e^J, \tag{3.3}$$

where we defined

$$J=-\frac{1}{2\hbar}\int_0^U du\sum_a\left[m\dot{x}_a^2(u)+\mu x_a^2(u)\right]$$
$$+\frac{1}{2\hbar^2}\int\int_0^U du\,du'\sum_{a,b}s_{ab}(u,u')x_a(u)x_b(u'). \tag{3.4}$$

The limit $n\to 0$ is to be taken. In Eq. (3.3) the variables $x_a(u)$ are scalars since a factor of N has been extracted. In the large-N limit the collective variables $r_{ab}(u,u')$ and $s_{ab}(u,u')$ are determined by the stationarity of the free energy. We will be looking for solutions of the saddle-point equations obeying translational invariance in the time direction, i.e., such that the order parameters depend only on the difference $\zeta = u-u'$. They should also be periodic functions of this variable with period U and also symmetric under $\zeta\to-\zeta$ (even functions).

In the case of a classical particle there is another approach to the problem called the variational approximation, which yields essentially the same results as the large-N limit aside from a simple renormalization of the correlation function of the disorder. For the quantum potential we were able to show that the same results still hold. Let us introduce the variational Hamiltonian

$$h_n = \frac{1}{2}\int_0^U du\sum_a\left[m\dot{\mathbf{x}}_a^2(u)+\mu\mathbf{x}_a^2(u)\right]$$
$$-\frac{1}{2\hbar}\int_0^U du\int_0^U du'\sum_{a,b}s_{ab}(u-u')\mathbf{x}_a(u)\cdot\mathbf{x}_b(u'). \tag{3.5}$$

Then the variational free energy is given by

$$n\beta\langle F\rangle_R/N = \langle\mathcal{H}_n-h_n\rangle_{h_n}/\hbar-\ln\int[dx]e^{-h_n/\hbar}. \tag{3.6}$$

In Appendix A we show that the variational free energy coincides with that given in the large-N limit [see Eq. (4.5) below], with the following renormalization of the function f defined in Eq. (2.6):

$$f(y)\to\hat{f}(y)=\frac{1}{\Gamma(N/2)}\int_0^\infty d\alpha\alpha^{N/2-1}e^{-\alpha}f\left(\frac{2\alpha}{N}y\right). \tag{3.7}$$

The effective correlation $\hat{f}$ is similar to f in the sense that for large arguments

$$\hat{f}(y)\sim\frac{\hat{g}}{2(1-\hat{\gamma})}y^{1-\hat{\gamma}}, \tag{3.8}$$

with

$$\hat{\gamma}=\begin{cases}\gamma \text{ if } \gamma\leq\frac{N}{2}+1\\ \frac{N}{2}+1 \text{ if } \gamma\geq\frac{N}{2}+1\end{cases}$$

and

$$\hat{g}=g\frac{\Gamma(1-\gamma+N/2)}{\Gamma(N/2)}\left(\frac{N}{2}\right)^{-1+\gamma}. \tag{3.9}$$

Since we will be mainly concerned with the case where $\hat{\gamma}=\gamma$, the values for g quoted in the numerical work below should actually pertain to the renormalized coupling $\hat{g}$. In the limit $N\to\infty$, $\hat{f}$ approaches f.

IV. THE STATIONARITY CONDITIONS AND THEIR EXACT SOLUTION ASSUMING REPLICA SYMMETRY

In this section we will derive the stationarity equations for the free energy and solve them in the replica-symmetric (RS) case. The case of RSB will be discussed in Sec. V. We proceed by transforming to frequency space by using a Fourier series representation of the order parameters

$$r_{ab}(\zeta) = \frac{1}{\beta}\sum_{l=-\infty}^{\infty} \exp(-i\omega_l\,\zeta)\,\tilde{r}_{ab}(\omega_l), \tag{4.1}$$

$$s_{ab}(\zeta) = \frac{1}{\beta}\sum_{l=-\infty}^{\infty} \exp(-i\omega_l\,\zeta)\,\tilde{s}_{ab}(\omega_l), \tag{4.2}$$

where

$$\omega_l = \frac{2\pi}{U}l, \quad l = 0, \pm1, \pm2, \dots \;. \tag{4.3}$$

Equation (4.3) follows from the periodicity of the order parameters. From the fact that $r_{ab}(\zeta)$ (and also s) are even functions, it follows that $\tilde{r}_{ab}(\omega) = \tilde{r}_{ab}(-\omega)$ and similarly for s. To proceed we note that for any periodic function $g(u - u')$ with period U it follows that

$$\begin{aligned}\int_0^U du \int_0^U du' g(u-u') &= \int_0^U du \int_0^u d\zeta g(\zeta) + \int_0^U du \int_{-U+u}^0 d\zeta g(\zeta)\\ &= \int_0^U du \int_0^u d\zeta g(\zeta) + \int_0^U du \int_u^U d\zeta g(\zeta - U)\\ &= \int_0^U du \int_0^U d\zeta g(\zeta) = U\int_0^U d\zeta g(\zeta).\end{aligned} \tag{4.4}$$

In terms of the new variables introduced in Eq. (4.2), the free energy becomes

$$\begin{aligned}n\beta\langle F\rangle_R/N &= \frac{1}{2}\sum_\omega\sum_{a,b}\tilde{r}_{ab}(\omega)\tilde{s}_{ab}(\omega) - \frac{1}{2}\sum_\omega \mathrm{tr}\;\ln G(\omega)\\ &+\frac{\beta}{2\hbar}\sum_{a,b}\int_0^U d\zeta f\left(\frac{1}{\beta}\sum_{\omega'}[\tilde{r}_{aa}(\omega') + \tilde{r}_{bb}(\omega')\right.\\ &\left.-2e^{-i\omega'\zeta}\,\tilde{r}_{ab}(\omega')]\right) + \mathrm{const},\end{aligned} \tag{4.5}$$

where we defined

$$G_{ab}(\omega) \equiv \{[(m\omega^2 + \mu)\mathbf{1} - \tilde{s}(\omega)]^{-1}\}_{ab}. \tag{4.6}$$

The constant in Eq. (4.5) can most easily be determined from the known free energy for a free particle ($f \equiv 0$ and $\mu = 0$).

From Eq. (4.5) it follows that stationarity equations are given by

$$\tilde{r}_{ab}(\omega) = G_{ab}(\omega), \tag{4.7}$$

$$\begin{aligned}\tilde{s}_{ab}(\omega) &= \frac{2}{\hbar}\int_0^U d\zeta e^{i\omega\zeta} f'\left(\frac{1}{\beta}\sum_{\omega'}[\tilde{r}_{aa}(\omega') + \tilde{r}_{bb}(\omega')\right.\\ &\left.-2e^{-i\omega'\zeta}\tilde{r}_{ab}(\omega')]\right), \quad a \neq b,\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\tilde{s}_{aa}(\omega) + \sum_{b\neq a}\tilde{s}_{ab}(0) + \frac{2}{\hbar}\int_0^U d\zeta\;(1 - e^{i\omega\zeta})\\ &\times f'\left(\frac{2}{\beta}\sum_{\omega'}\tilde{r}_{aa}(\omega')\;(1 - e^{-i\omega'\zeta})\right) = 0.\end{aligned} \tag{4.9}$$

In Eq. (4.8), f' stands for the first derivative of the function f, characterizing the correlations of the random potential.

We proceed to look for a RS solution to these equations by taking all the elements $\tilde{r}_{ab}(\omega) \equiv \tilde{r}(\omega)$ with $a \neq b$ to be equal to each other (for the same frequency) and similarly for $\tilde{s}_{ab}(\omega) \equiv \tilde{s}(\omega)$. The diagonal elements are also taken to be equal to each other and are denoted by $\tilde{r}_d(\omega)$ and $\tilde{s}_d(\omega)$. In the limit $n \to 0$ we find from Eqs. (4.9) and (4.8) by inverting an $n \times n$ matrix,

$$\begin{aligned}\tilde{r}_d(\omega) &= \frac{1}{m\omega^2 + \mu - \tilde{s}_d(\omega) + \tilde{s}(\omega)}\\ &+\frac{\tilde{s}(\omega)}{[m\omega^2 + \mu - \tilde{s}_d(\omega) + \tilde{s}(\omega)]^2},\end{aligned} \tag{4.10}$$

$$\tilde{r}(\omega) = \frac{\tilde{s}(\omega)}{[m\omega^2 + \mu - \tilde{s}_d(\omega) + \tilde{s}(\omega)]^2}. \tag{4.11}$$

These expressions have to be substituted into Eqs. (4.8) and (4.9) to obtain the self-consistent equations for $\tilde{s}$ and $\tilde{s}_d$:

$$\begin{aligned}\tilde{s}(\omega) &= \frac{2}{\hbar}\int_0^U d\zeta\; e^{i\omega\zeta}\\ &\times f'\left[\frac{2}{\beta}\sum_{\omega'}\left(\frac{1}{m\omega'^2 + \mu - \tilde{s}_d(\omega') + \tilde{s}(\omega')}\right.\right.\\ &\left.\left.+\frac{\tilde{s}(\omega')\;(1 - e^{-i\omega'\zeta})}{[m\omega^2 + \mu - \tilde{s}_d(\omega) + \tilde{s}(\omega)]^2}\right)\right],\end{aligned} \tag{4.12}$$

$$\begin{aligned}\tilde{s}_d(\omega) &= \tilde{s}(0) - \frac{2}{\hbar}\int_0^U d\zeta\;(1 - e^{i\omega\zeta})\\ &\times f'\left[\frac{2}{\beta}\sum_{\omega'}(1 - e^{-i\omega'\zeta})\right.\\ &\times\left(\frac{1}{m\omega'^2 + \mu - \tilde{s}_d(\omega') + \tilde{s}(\omega')}\right.\\ &\left.\left.+\frac{\tilde{s}(\omega')}{[m\omega'^2 + \mu - \tilde{s}_d(\omega') + \tilde{s}(\omega')]^2}\right)\right].\end{aligned} \tag{4.13}$$

We will now seek a solution with time-independent off-diagonal elements. This is in analogy with quantum spin glasses [16,19], the rationale being that the off-diagonal elements of r and s are the glass order parameters and hence are constants independent of the Trotter time. We find that such a solution satisfies the saddle-point equations exactly. The diagonal elements are still time dependent. In frequency space we look for solutions of the form

$$\tilde{s}(\omega) = \tilde{s}\;\delta_{\omega,0}. \tag{4.14}$$

This ansatz satisfies Eq. (4.12) and we obtain

$$\tilde{s} = 2\beta f'\left(\frac{2}{\beta\mu} + \frac{2}{\beta}\sum_{\omega\neq 0}\frac{1}{m\omega'^2 + \mu - \tilde{s}_d(\omega')}\right), \tag{4.15}$$

$$\tilde{s}_d(\omega) = \tilde{s} - \frac{2}{\hbar}\int_0^U d\zeta\,(1 - e^{i\omega\zeta}) \times f'\left(\frac{2}{\beta}\sum_{\omega'\neq 0}(1 - e^{-i\omega'\zeta})\frac{1}{m\omega'^2 + \mu - \tilde{s}_d(\omega')}\right). \tag{4.16}$$

Two physical quantities of interest are

$$\langle\langle \mathbf{x}^2\rangle\rangle_R/N = \frac{1}{n}\sum_{a=1}^{n} r_{aa}(0) = r_d(0) = \frac{1}{\beta}\sum_{k=-\infty}^{\infty}\tilde{r}_d(\omega_k), \tag{4.17}$$

$$\langle\langle \mathbf{x}^2\rangle - \langle\mathbf{x}\rangle^2\rangle_R/N = \frac{1}{n}\sum_{a=1}^{n} r_{aa}(0) - \frac{1}{n(n-1)}\sum_{a\neq b}^{n} r_{ab}(0) = r_d(0) - r(0). \tag{4.18}$$

The very last equality holds if one assumes replica symmetry

$$\langle\langle \mathbf{x}^2\rangle\rangle_R/N = \frac{1}{\beta\mu} + \frac{1}{\beta}\sum_{\omega\neq 0}\frac{1}{m\omega^2 + \mu - \tilde{s}_d(\omega)} + \frac{\tilde{s}}{\beta\mu^2} \tag{4.19}$$

and

$$\langle\langle \mathbf{x}^2\rangle - \langle\mathbf{x}\rangle^2\rangle_R/N = \frac{1}{\beta\mu} + \frac{1}{\beta}\sum_{\omega\neq 0}\frac{1}{m\omega^2 + \mu - \tilde{s}_d(\omega)}. \tag{4.20}$$

In all the equations derived above, m and $\hbar$ enter only through the combination

$$\kappa \equiv \frac{m}{\hbar^2}\ . \tag{4.21}$$

The classical limit is given by $\hbar \to 0$ and consequently $\kappa \to \infty$. Practically, this limit can be achieved by taking the particle's mass to be very large. On the other hand, the quantum limit is obtained for very small particle mass. In the classical limit only the zero-frequency component of the observables survives and we obtain, e.g.,

$$\langle\langle x^2\rangle\rangle_R = \frac{1}{\beta\mu} + \frac{2}{\mu^2}f'\left(\frac{2}{\beta\mu}\right), \tag{4.22}$$

which agrees with Refs. [8,11].

From Eqs. (4.19) and (4.15) it follows that the expression for $\langle\langle x^2\rangle\rangle_R$ in the quantum case is the same as in the classical case, but with a renormalized temperature $1/\beta_R$ given by

$$\frac{1}{\mu\beta_R} = \frac{1}{\mu\beta} + \frac{1}{\beta}\sum_{\omega\neq 0}\frac{1}{m\omega'^2 + \mu - \tilde{s}_d(\omega')} = \frac{1}{\mu\beta} + \frac{1}{2}b_0(\beta,\kappa,\mu,g), \tag{4.23}$$

where we defined

$$b_0(\beta,\kappa,\mu,g) = \frac{2}{\beta}\sum_{\omega\neq 0}\frac{1}{m\omega^2 + \mu - \tilde{s}_d(\omega)}. \tag{4.24}$$

The minimum of the mean-square displacement in Eq. (4.22) is attained for

$$T_c^{\mathrm{cl}} = 1/\beta_c^{\mathrm{cl}} = \frac{1}{2}(2\gamma\, g)^{1/(1+\gamma)}\mu^{-(1-\gamma)/(1+\gamma)} - \frac{1}{2}a_0\mu. \tag{4.25}$$

In the quantum case, it follows from Eq. (4.23) that the temperature for which the minimum is attained is given by a solution of the equation

$$T_c + \frac{1}{2}\mu b_0(1/T_c,\kappa,\mu,g) = T_c^{\mathrm{cl}}. \tag{4.26}$$

We will see in Sec. V that for the case of long-range correlations, the temperature for which the minimum of the mean-square displacement is attained is indeed the transition temperature into the glassy state characterized by RSB. For the case of short-range correlation of the disorder, even in the classical case the transition into a replica-broken phase occurs only approximately at the temperature given by Eq. (4.25) if μ is not too small [11]. Otherwise the transition T^* into a RSB phase occurs at a higher temperature than given by (4.25). The value of the mean-square displacement at the minimum is given by

$$\langle\langle \mathbf{x}^2\rangle\rangle_R/N = (1+\gamma)(2\gamma)^{-\gamma/(1+\gamma)}g^{1/(1+\gamma)}\mu^{-2/(1+\gamma)} - \frac{1}{2}a_0. \tag{4.27}$$

We now turn to the numerical solution of Eq. (4.16). Some technical details of the numerical procedure are given in Appendix B. First we consider the case of long-range correlations of the disorder with $\gamma = 1/2$. In Fig. 2 we display the results for the mean-square displacement for various values of $\kappa = m/\hbar^2$. We have chosen $g = 2\sqrt{2}$, $a_0 = 0.01$, and $\mu = 1$ in order to compare with results of the classical case [8]. For these values of the parameters, $T_c(\infty) = 0.995$. For large κ there is good agreement with known results for the classical particle [8]. As κ decreases and quantum effects become more significant, the minimum of the curves moves to the left, which shows that b_0 is increasing and hence T_c is decreasing [see Eq. (4.26)]. For $\kappa \approx 0.17$ the minimum of the curve is at about $T = 0$ and for lower values of κ no minimum occurs at a positive temperature. From this data we can reconstruct an approximate phase diagram in the $T-1/\kappa$ plane; see Fig. 3.

We now turn to the case of short-range correlations of the disorder, i.e., $\gamma = 3/2$ [recall Eq. (2.6)]. We have

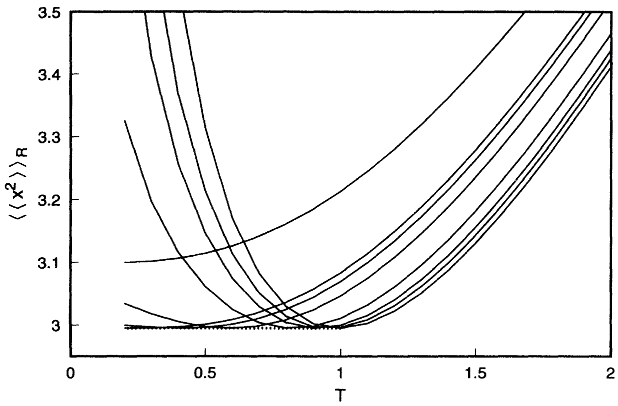

FIG. 2. Plot of the mean-square displacement vs $t = T/T_c(\infty)$ for the long-range case ($\gamma = 1/2$). Curves from right to left are for $\kappa = 1000, 2, 1, 0.5, 0.25, 0.2, 0.18, 0.1$. The horizontal dashed line represents the RSB solution below $T_c(\kappa)$.

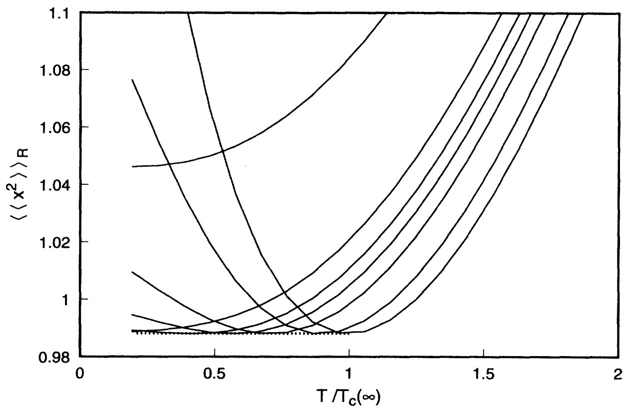

FIG. 4. Plot of the mean-square displacement vs T for the short-range case ($\gamma = 3/2$). The notation is the same as in Fig. 2.

chosen $a_0 = 1$, $g = 1/\sqrt{2\pi}$, and $\mu = 0.53$ in order to compare with the results of the classical case [11]. In Fig. 4 we plot the mean-square displacement given by Eq. (4.19) for various values of κ as a function of $t = T/T_c(\infty)$, where $T_c(\kappa)$ is the temperature for which the "transition" into the glassy state occurs. Thus $t = 1$ is the rescaled temperature for which the transition occurs in the classical case ($\kappa = \infty$). For the values of the parameters we have chosen, $T_c(\infty) = 0.2082$. We see that the situation is similar to the long-range case. The value of κ for which the minimum of the curve hits zero is about 2. The reconstructed phase diagram is given in Fig. 5.

In Appendix C we review, correct, and present more details of the real-space approach used in our previous investigation [13]. The formalism is set up for first-step RSB. (The need for RSB at low temperature will be discussed in Sec. V.) We have checked numerically that for replica symmetry and long-ranged correlations of the disorder the plots of the mean-square displacement vs temperature coincide with those obtained from the momentum space calculations as reported above.

We finish this section with a discussion of the zero-temperature limit. In this limit the frequency becomes a continuous variable and the equations for $\tilde{s}$ and $\tilde{s}_d(\omega)$ become

$$\tilde{s}/\beta = 2f'\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{\kappa\omega'^2+\mu-\tilde{s}_d(\omega'/\hbar)}\right),$$

$$\tilde{s}_d(\omega/\hbar) = -2\int_{-\infty}^{\infty}d\zeta(1-e^{i\omega\zeta}) \times\left[f'\left(\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega'\frac{1-e^{-i\omega'\zeta}}{\kappa\omega'^2+\mu-\tilde{s}_d(\omega'/\hbar)}\right) -f'\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{\kappa\omega'^2+\mu-\tilde{s}_d(\omega'/\hbar)}\right)\right],\quad \omega\neq 0$$

$$\tilde{s}_d(0) = \tilde{s}. \tag{4.28}$$

where we have rescaled the variables ζ and ω by $1/\hbar$ and $\hbar$ respectively. In terms of the solution to Eq. (4.28), the expression for the mean-square displacement becomes

$$\langle\langle \mathbf{x}^2\rangle\rangle_R/N = \frac{1}{2}b_0(\beta=\infty,\kappa,\mu,g) +2f'(b_0(\beta=\infty,\kappa,\mu,g)). \tag{4.29}$$

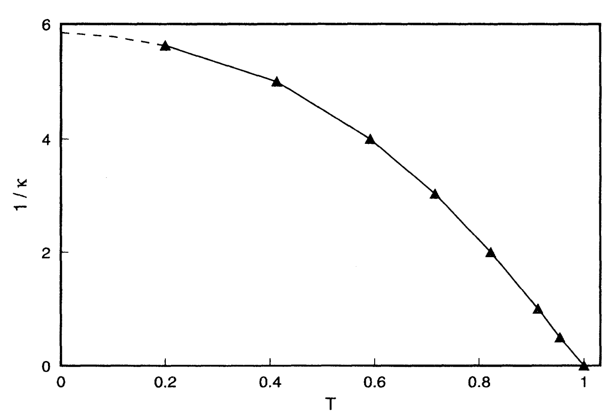

FIG. 3. Phase diagram for the long-range case for $g = 2\sqrt{2}$ and $\mu = 1$. The dashed line is a possible extrapolation to $T = 0$.

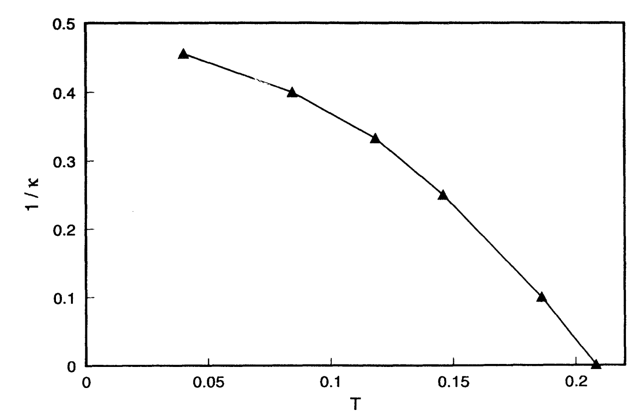

FIG. 5. Phase diagram for the short-range case for $g = 1/\sqrt{2\pi}$ and $\mu = 0.53$. Curves from right to left are for $\kappa = 1000, 10, 4, 3, 2.5, 2, 1$.

with

$$b_0(\beta=\infty,\kappa,\mu,g) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega'}{\kappa\omega'^2+\mu-\tilde{s}_d(\omega'/\hbar)}. \quad (4.30)$$

Equation (4.29) is similar to the classical expression (4.19), with the temperature variable T replaced by $(1/2)\mu b_0(\beta=\infty,\kappa,\mu,g)$. From Eq. (4.26) it follows that the quantum transition at $T=0$ (for the case of long-ranged correlations) occurs when $\kappa=\kappa_c$, where

$$\frac{1}{2}b_0(\beta=\infty,\kappa_c,\mu,g) = \frac{T_c^{\rm cl}}{\mu}. \quad (4.31)$$

V. REPLICA-SYMMETRY-BREAKING SOLUTION

In this section we proceed to explore the possibility of RSB. The need to break replica symmetry has been demonstrated in the classical case below a certain transition temperature T_c. The physical origin of RSB is the existence of many local minima of the free energy, separated by barriers. A sharp transition temperature for a single particle exists only in the large-N limit or within the framework of the variational (Gaussian) approximation. Classically the pattern of RSB depends on the range of correlations of the random potential. For short-range correlations a one-step RSB has been found sufficient, whereas for long-range correlations a continuous RSB has been found necessary [8,11]. In the quantum case we will show below that RSB also occurs if the strength of the quantum fluctuations is not too large. Support for this assertion also comes from the fact that for the RS solution a plot of $\langle\langle x^2\rangle\rangle_R$ vs temperature (Fig. 2) is not monotonic, but increases as $T\to 0$ for κ not too small. This indicates that the RS solution is inadequate at low temperatures in this case. Practically one can compare the free energies associated with the RS and RSB solutions (if the latter exists) and verify that the free energy of the RSB solution is *higher* (an artifact of the $n\to 0$ limit). This indicates that RS has to be broken. This is indeed the situation in the classical case and it carries over to the quantum case as explained below.

In constructing a RSB solution we associate

$$\tilde{s}_{aa}(\omega) = \tilde{s}_d(\omega), \quad (5.1)$$

$$\tilde{s}_{ab}(\omega,z) \leftrightarrow \tilde{s}(z)\,\delta_{\omega,0}, \quad a\neq b \quad (5.2)$$

where the Parisi parameter $0<z<1$ labels the "distance" between the replica indices ab. Again, we have used the static ansatz for the nondiagonal matrix elements, which will turn out to be a consistent solution. One should not confuse the frequency dependence of the diagonal element $\tilde{s}_d(\omega)$ with the z-parameter dependence of $\tilde{s}(z)$. A similar parametrization applies to $\tilde{r}_{ab}(\omega)$. To write down the expressions for $\tilde{r}$, analogous to Eq. (4.11), we need the expression for the inverse of a hierarchical matrix (see Appendix II of [8]). We then find

$$\tilde{r}_d(\omega) = \frac{1}{m\omega^2+\mu-\tilde{s}_d(\omega)+\delta_{\omega,0}\int_0^1 dz\tilde{s}(z)} \times\left(1+\frac{\tilde{s}(z=0)\delta_{\omega,0}}{\mu} +\delta_{\omega,0}\int_0^1 \frac{dz}{z^2}\,\frac{[\tilde{s}](z)}{\mu+[\tilde{s}](z)}\right), \quad (5.3)$$

$$\tilde{r}(\omega,z) = \tilde{r}(z)\delta_{\omega,0} = \delta_{\omega,0}\,\frac{1}{\mu}\left(\frac{[\tilde{s}](z)}{z(\mu+[\tilde{s}](z))}+\frac{\tilde{s}(z=0)}{\mu} +\int_0^z \frac{dz}{z^2}\,\frac{[\tilde{s}](z)}{\mu+[\tilde{s}](z)}\right), \quad (5.4)$$

where

$$[\tilde{s}](z) = z\,\tilde{s}(z) - \int_0^z dz\,\tilde{s}(z). \quad (5.5)$$

Using these formulas, the stationarity equations [see Eqs. (4.8) and (4.9)] become

$$\tilde{s}(z) = 2\beta f'\left[\frac{2}{\beta}\left(\frac{1}{z(\mu+[\tilde{s}](z))} - \int_z^1 \frac{dz}{z^2}\,\frac{1}{\mu+[\tilde{s}](z)}\right) +\frac{2}{\beta}\sum_{\omega\neq 0}\frac{1}{m\omega^2+\mu-\tilde{s}_d(\omega)}\right], \quad (5.6)$$

$$\tilde{s}_d(\omega) = \int_0^1 dz\,\tilde{s}(z) - \frac{2}{\hbar}\int_0^U d\zeta\,(1-e^{i\omega\zeta}) \times f'\left(\frac{2}{\beta}\sum_{\omega'\neq 0}(1-e^{-i\omega'\zeta})\frac{1}{m\omega'^2+\mu-\tilde{s}_d(\omega')}\right). \quad (5.7)$$

If we compare Eq. (5.6) with the corresponding equation in the classical case, we see that it is the same, apart from a "renormalization" or shift of the constant a_0 appearing in the definition of the function f [see Eq. (2.6)]:

$$a_0 \to a_R(\beta,\kappa,\mu,g) = a_0 + b_0(\beta,\kappa,\mu,g). \quad (5.8)$$

The crucial point is that b_0 is independent of z. We will discuss first the case of long-range correlations of the disorder. One can repeat the steps carried out for the classical case [8,11] and we find the following solution to Eq. (5.6):

$$\tilde{s}(z) = \begin{cases} \frac{3}{2}Az_1^2, & 0<z<z_1 \\ \frac{3}{2}Az^2, & z_1<z<z_2 \\ \frac{3}{2}Az_2^2, & z_2<z<1, \end{cases}$$

with

$$A = (2/3)^3 g^2\beta^3, \quad (5.9)$$

$$z_1 = \frac{3}{2}g^{-2/3}\mu^{1/3}\beta^{-1}, \quad (5.10)$$

and z_2 is the solution of the equation

$$\frac{1}{2}\beta A a_R z_2^4 + z_2 - \frac{3}{4} = 0 \,. \tag{5.11}$$

We observe that the only difference from the classical case is the appearance of the renormalized function a_R instead of a_0 in Eq. (5.11). This solution becomes replica symmetric for $z_1 = z_2$, which occurs when

$$T_c = 1/\beta_c = \frac{1}{2} g^{2/3}\mu^{-1/3} - \frac{1}{2} a_R(\beta_c, \kappa, \mu, g). \tag{5.12}$$

This constitutes an equation for the transition temperature into the glassy phase. Notice that for $\gamma = 1/2$ this is the same as Eq. (4.26) in Sec. IV, which was obtained by minimizing the mean-square displacement in the replica symmetric case. Also, it follows from Eqs. (4.17) and (5.3) that in the presence of RSB

$$\langle\langle \mathbf{x}^2\rangle\rangle_R/N = \frac{1}{\beta\mu} + \frac{\tilde{s}(z=0)}{\beta\mu^2} + \frac{1}{\beta\mu}\int_0^1 \frac{dz}{z^2}\frac{[\tilde{s}](z)}{\mu + [\tilde{s}](z)} + \frac{1}{\beta}\sum_{\omega\neq 0}\frac{1}{m\omega^2 + \mu - \tilde{s}_d(\omega)}. \tag{5.13}$$

Using the RSB solution obtained above, we find

$$\begin{aligned}\langle\langle \mathbf{x}^2\rangle\rangle_R/N &= \frac{3}{2} g^{2/3}\mu^{-1/3} - \frac{1}{2} a_R + \frac{1}{2} b_0 \\ &= \frac{3}{2} g^{2/3}\mu^{-1/3} - \frac{1}{2} a_0,\end{aligned} \tag{5.14}$$

which is exactly the same as found for the minimum of the replica-symmetric expression in Sec. IV [Eq. (4.27) for $\gamma = 1/2$]. Thus, through the entire glassy phase the mean-square displacement remains locked at this constant value.

Let us comment briefly on the zero-temperature limit of the RSB solution. In that limit the solution given in Eq. (5.9) becomes

$$\frac{\tilde{s}(z)}{\beta} = \begin{cases} (g\mu)^{2/3}, & 0 < z < z_1 \\ (\frac{2}{3})^2 (g\beta z)^2, & z_1 < z < z_2 \\ g a_R^{-1/2}, & z_2 < z < 1, \end{cases}$$

with

$$z_1 = \frac{3}{2} g^{-2/3}\mu^{1/3}\beta^{-1}, \tag{5.15}$$

$$z_2 = \frac{3}{2} g^{-1/2} a_R^{-1/4}\beta^{-1}. \tag{5.16}$$

For $\beta \to \infty$ this solution appears more replica-symmetric-like, but it really is not, since the value at $z = 0$ is always different from the value at $z \neq 0$ and the contribution to the mean-square displacement remains flat.

In the short-range case Eq. (5.6) has a one-step RSB solution, as in the classical case, again since the only difference from the classical equation is a renormalization of a_0. The equations can only be solved numerically [11] and a detailed investigation will be carried out elsewhere.

VI. CONCLUDING REMARKS

In this paper we have investigated the combined effects of quantum fluctuations and quenched disorder for the case of a particle subjected to a combination of a fixed harmonic potential and a random potential with power law spatial correlations. Acting separately, disorder and quantum fluctuations both increase the mean-square displacement of the particle. From Figs. 2 and 4 it becomes evident that at high temperature, adding quantum fluctuations in addition to disorder increases the mean-square displacement, but at low temperature, in the phase characterized by glassy behavior, the mean-square displacement remains locked probably due to tunneling effects among the different free-energy minima.

We have seen that the phase diagram of the model is similar to that of the quantum spin glass in a transverse field. There is a transition at zero temperature from a phase characterized by RSB effects to a phase where such effects are not present.

It may be possible to extend the problem of a quantum particle to higher-dimensional manifolds in a disordered medium and explore the effects of quantum fluctuations. It may be also possible to extend the problem to many particle systems where statistics might play an important role. We hope our work will stimulate further research in this interesting area.

ACKNOWLEDGMENTS

I thank Dr. H. A. Duncan for useful discussions and Hsuan-Yi Chen for checking some of the algebra and performing the numerical calculation reported in Appendix C.

APPENDIX A

The major step in calculating the variational free energy is evaluating the expectation value

$$\left\langle f\left(\frac{[\mathbf{x}_a(u) - \mathbf{x}_b(u')]^2}{N}\right)\right\rangle_{h_n}. \tag{A1}$$

This is done by expanding f in a power series and using the formula

$$\langle[\mathbf{x}_a(u) - \mathbf{x}_b(u')]^{2j}\rangle_{h_n} = \frac{(N+2j-2)!!}{(N-2)!!}[\hat{G}_{aa}(0) + \hat{G}_{bb}(0) - 2\hat{G}_{ab}(u-u')]^j \,, \tag{A2}$$

where we defined

$$N\hat{G}_{ab}(u-u') = \langle \mathbf{x}_a(u)\cdot\mathbf{x}_b(u')\rangle_{h_n}. \tag{A3}$$

Resumming the series gives rise to $\hat{f}(\hat{G}_{aa}(0) + \hat{G}_{bb}(0) -$

$2\hat{G}_{ab}(u-u'))$.

In momentum space conjugate to the Trotter time the variational free energy becomes

$$n\beta\langle F\rangle_R/N = \frac{1}{2}\sum_{\omega}\sum_{ab} G_{ab}(\omega)\tilde{s}_{ab}(\omega) - \frac{1}{2}\sum_{\omega}\mathrm{tr}\ \ln G(\omega) + \frac{\beta}{2\hbar}\sum_{a,b}\int_0^U d\zeta \hat{f}\left(\frac{1}{\beta}\sum_{\omega'}[G_{aa}(\omega') + G_{bb}(\omega') - 2e^{-i\omega'\zeta}\ G_{ab}(\omega')]\right) + \text{const}, \tag{A4}$$

where $\tilde{s}_{ab}(\omega)/\beta$ is the Fourier transform of $s_{ab}(u-u')$ and $G_{ab}(\omega)$ is the Fourier transform of $\hat{G}_{ab}(u-u')$. This free energy coincides with the one derived from the large-N limit with the replacement $f\to\hat{f}$.

APPENDIX B

In this appendix we provide some further details on the numerical solution of Eq. (4.16) and the evaluation of the mean-square displacement Eq. (4.19). Our goal is to minimize the error in the truncation of high frequencies. Hence we define the function $t_d(\omega)$ for $\omega\neq 0$ by

$$\tilde{t}_d(\omega_l) = \tilde{s}_d(\omega) + \tilde{t}, \tag{B1}$$

where $\tilde{t}$ will be adjusted such that for the highest frequency included $\tilde{t}_d(\omega_{\max}) = 0$. We then use the formula [20]

$$\frac{1}{\beta}\sum_{\omega\neq 0}\frac{e^{-i\omega\zeta}}{m\omega^2+\mu} = \frac{1}{2\sqrt{\kappa\mu}}\frac{\cosh[\alpha(1-2\zeta/U)]}{\sinh(\alpha)} - \frac{1}{\beta\mu}, \tag{B2}$$

with

$$\alpha = \frac{\beta}{2}\sqrt{\frac{\mu}{\kappa}}, \tag{B3}$$

to write

$$\frac{2}{\beta}\sum_{\omega\neq 0}\frac{e^{-i\omega\zeta}}{m\omega^2+\mu-\tilde{s}_d(\omega)} = -\frac{2}{\beta\mu_R} + \frac{1}{\sqrt{\kappa\mu_R}}\frac{\cosh[\alpha_R(1-2\zeta/U)]}{\sinh(\alpha_R)} + \frac{4}{\beta}\sum_{k=1}^{k_m}\frac{e^{-i\omega_k\zeta}\tilde{t}_d(\omega_k)}{(m\omega_k^2+\mu_R)(m\omega_k^2+\mu_R-\tilde{t}_d(\omega_k))}, \tag{B4}$$

where

$$\mu_R = \mu + \tilde{t}, \tag{B5}$$

$$\alpha_R = \frac{\beta}{2}\sqrt{\frac{\mu_R}{\kappa}}. \tag{B6}$$

A similar expression is used for the case of $\zeta = 0$. The equations for $\tilde{t}_d(\omega_k)$, $\tilde{t}$, and $\tilde{s}$ were solved numerically with $k_{\max} = 10$. We used a FORTRAN routine [21] that finds a root for a set of nonlinear equations. The solution was then used to calculate the mean-square displacement.

APPENDIX C

In this appendix we give some of the details omitted in Ref. [13] for lack of space, as well implement some corrections. Some of the notation is slightly different than in the present paper. Starting with Eq. (3.3) we proceeded in I to look for a saddle point under the constraint that the off-diagonal elements of the order parameters r and s are independent of the Trotter time, whereas the diagonal elements denoted by $\chi(u-u')\equiv r_d(u-u')$ and $\nu(u-u')\equiv s_d(u-u')$ are dependent on $u-u'$. In the one-step RSB we have used the notation

$$r(z) = r_2,\quad z<k \tag{C1}$$

$$r(z) = r_{11},\quad z>k, \tag{C2}$$

where k has been used to denote the breaking point z_c. A similar notation applies for s. Substituting these order parameters in the expression for the free energy we obtain

$$\beta\langle F\rangle_R/N = \frac{\beta^2}{2}[k(r_2s_2 - r_{11}s_{11}) - r_2s_2] + \frac{1}{2\hbar^2}\int_0^U\int_0^U dudu'\chi(u-u')\nu(u-u') - \frac{\beta^2}{2}kf(2\chi(0)-2r_{11}) + \frac{\beta^2}{2\hbar^2}(k-1)f(2\chi(0)-2r_2) + \frac{1}{2\hbar^2}\int\int dudu' f(2\chi(0)-2\chi(u-u')) - \frac{1}{k}\int_{-\infty}^{\infty}Dz\ \ln\int Dy\ (Z_H)^k + \text{const}, \tag{C3}$$

where $Dz \equiv (dz/\sqrt{2\pi})\exp(-z^2/2)$ and similarly for Dy, and $Z_H\equiv\int dx\int[dx]e^{-H/\hbar}$. H is the one-dimensional, one-particle effective Hamiltonian given by

$$H = \int_0^U du\left(\frac{m}{2}\dot{x}^2(u) + \frac{\mu}{2}x^2(u) - (z\sqrt{s_{11}} + y\sqrt{s_2 - s_{11}})x(u)\right) - \frac{1}{2\hbar}\int_0^U\int_0^U dudu'\ [\nu(u-u') - s_2]\,x(u)x(u'). \tag{C4}$$

The different order parameters are determined self-consistently from the equations that extremize $\langle F\rangle_R$:

$$\chi(u-u') = \int Dz\frac{\int Dy\,\langle x(u)x(u')\rangle_H\,Z_H^k}{\int DyZ_H^k}, \tag{C5}$$

$$r_{11} = \int Dz\left(\frac{\int Dy\,\langle x(u)\rangle_H\,Z_H^k}{\int DyZ_H^k}\right)^2, \tag{C6}$$

$$r_2 = \int Dz \frac{\int Dy \langle x(u)\rangle_H^2 Z_H^k}{\int Dy Z_H^k}, \tag{C7}$$

$$\begin{aligned}\nu(u-u') = {} & [2kf'(2\chi(0)-2r_{11}) \\ & +2(1-k)f'(2\chi(0)-2r_2) \\ & -\frac{2}{U^2}\int_0^U\int_0^U dudu' f'(2\chi(0)-2\chi(u-u'))] \\ & \times U\delta(u-u') + 2f'(2\chi(0)-2\chi(u-u')),\end{aligned} \tag{C8}$$

$$s_{11} = 2f'(2\chi(0)-2r_{11}), \tag{C9}$$

$$s_2 = 2f'(2\chi(0)-2r_2). \tag{C10}$$

The equation for the breaking point k is given by

$$\begin{aligned}&\frac{\beta^2}{2}(r_2 s_2 - r_{11}s_{11}) - \frac{\beta^2}{2} f(2\chi(0)-2r_{11}) \\ &\quad + \frac{\beta^2}{2} f(2\chi(0)-2r_2) + \frac{1}{k^2}\int_{-\infty}^{\infty} Dz \ln \int Dy (Z_H)^k \\ &\quad - \frac{1}{k}\int Dz \frac{\int Dy (Z_H)^k \ln Z_H}{\int Dy (Z_H)^k} = 0. \end{aligned} \tag{C11}$$

We have continued by putting the time variable on a lattice, with lattice spacing U/M, where M is to be taken eventually to infinity. In practice we carried out exact calculations up to values of $M = 20$ and extrapolation to $M = \infty$ has been made by finite-size scaling and the $1/M^2$ rule [16]. The calculation involves finding the inverse and the determinant of the $M \times M$ matrix $\mathcal{M}$ appearing in the expression for the discretized effective Hamiltonian; see Eq. (C4):

$$\mathcal{M}_{i,j} = \delta_{ij}\left(2\kappa\frac{M}{\beta} + \frac{\mu\beta}{M}\right) - \frac{\kappa M}{\beta}\delta_{i+1,j} - \frac{\beta^2}{M^2}[\nu(|i-j|) - s_2], \quad i \le j, i \ne 1, \tag{C12}$$

where $\kappa \equiv m/\hbar^2$. For $i = 1$ there is an additional term $(-\kappa M/\beta)\delta_{M,j}$. For $j < i$ the matrix is given by symmetry.

Using this notation, the final result for the discretized free energy is

$$\begin{aligned}\beta\langle F\rangle_R/N = {} & \frac{\beta^2}{2}[k(r_2s_2 - r_{11}s_{11}) - r_2s_2] + \frac{\beta^2}{2M^2}\sum_{i,j}\chi(|i-j|)\nu(|i-j|) \\ & -\frac{k\beta^2}{2}f(2\chi(0)-2r_{11}) + \frac{(k-1)\beta^2}{2}f(2\chi(0)-2r_2) + \frac{\beta^2}{2M^2}\sum_{i,j} f(2\chi(0)-2\chi(|i-j|)) \\ & -\frac{M}{2}\ln\left(\frac{\kappa M}{\beta}\right) + \frac{1}{2}\ln\ \det\ \mathcal{M} + \frac{1}{2k}\ln(1-k\sigma v_2^2) - \frac{\sigma v_1^2}{2(1-k\sigma v_2^2)},\end{aligned} \tag{C13}$$

where we defined

$$\sigma = \frac{1}{M}\sum_j (\mathcal{M}^{-1})_{ij}, \quad v_1^2 = \beta^2 s_{11}, \quad v_2^2 = \beta^2(s_2 - s_{11}). \tag{C14}$$

The different order parameters have been obtained by extremizing Eq. (C13).

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